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LETTER TO THE EDITOR

# Matrix-bipolar asymptotic modules for solving (2+1)-dimensional non-linear evolution equations with constraints

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**Abstract.** An infinite hierarchy of solvable systems of purely differential non-linear equations is introduced within the framework of asymptotic modules. Each system consists of (2+1)-dimensional evolution equations for at most four complex functions and of quite strong differential constraints. It may be interpreted formally as an integro-differential equation in (1+1) dimensions.

This letter is part of a series on a new method for introducing and solving non-linear equations (NE). Among the NE investigated here, we mention the systems ($1a^+$, $b^+$, $c^+$) and ($1a^-$, $b^-$, $c^-$) where

$$q_t = \alpha_3(\tfrac{1}{4}q_{xxx} \mp \tfrac{3}{2}q^2 q_x) + \alpha_{-3}(-\tfrac{1}{4}r_{xyy} \pm \tfrac{3}{2}r^2 r_x) \qquad (1a^\pm)$$

$$r_t = \alpha_{-3}(\tfrac{1}{4}r_{yyy} \mp \tfrac{3}{2}r^2 r_y) + \alpha_3(-\tfrac{1}{4}q_{yxx} \pm \tfrac{3}{2}q^2 q_y) \qquad (1b^\pm)$$

$$q_y = -r_x \qquad \frac{q_{xy}}{q} = \frac{r_{xy}}{r} \qquad \left(\frac{q_{xy}}{4q}\right)^2 = 1 \pm \tfrac{1}{4}(q_y)^2 \qquad (1c^\pm)$$

and where the unknowns are the complex functions $q(x, y, t)$ and $r(x, y, t)$, $\alpha_3$ and $\alpha_{-3}$ are given complex numbers and $q_t$, $q_x, \ldots,$ mean $\partial_t q$, $\partial_x q, \ldots.$ Observe that ($1a^\pm$) and ($1b^\pm$) are evolution equations for $q$, $r$ and ($1c^\pm$) are differential constraints. In fact, since ($1c^\pm$) implies $r = -qq_{yy}/q_{xy}$, it is easy to see that $q$ satisfies the system consisting of the evolution equation

$$q_t = \alpha_3(\tfrac{1}{4}q_{xxx} \mp \tfrac{3}{2}q^2 q_x) + \alpha_{-3}[\tfrac{1}{4}q_{yyy} \mp \tfrac{3}{2}q^2 q_y(q_{yy}/q_{xy})^2]$$

and of the constraint $(q_{xy}/4q)^2 = 1 \pm \frac{1}{4}(q_y)^2$. Note also that for $\alpha_3 = -4$ and $\alpha_{-3} = 0$, ($1a^\pm$) reduces to the modified Korteweg-de Vries (MKdV) equation $q_t + q_{xxx} \mp 6q^2 q_x = 0$ relative to the variables $(x, t)$.

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We also mention the systems $(2a^+, b^+, c^+)$ and $(2a^-, b^-, c^-)$ where

$$q_t = -\mathrm{i}\alpha_2(\tfrac{1}{2}q_{xx} \mp q|q|^2) - \mathrm{i}\alpha_{-2}\frac{q_y}{r_x}(-r+\tfrac{1}{4}r_{xy}) + \mathrm{i}\alpha_{-2}\frac{q_y}{\bar{r}_x}(\bar{r}+\tfrac{1}{4}\bar{r}_{xy}) \qquad (2a^\pm)$$

$$r_t = -\mathrm{i}\alpha_{-2}(\tfrac{1}{2}r_{yy} \mp r|r|^2) - \mathrm{i}\alpha_2\frac{r_x}{q_y}(-q+\tfrac{1}{4}q_{xy}) + \mathrm{i}\alpha_2\frac{r_x}{\bar{q}_y}(\bar{q}+\tfrac{1}{4}\bar{q}_{xy}) \qquad (2b^\pm)$$

$$|q_y|^2 = |r_x|^2 \qquad \frac{q_{xy}}{q} = \frac{\bar{q}_{xy}}{\bar{q}} = \frac{r_{xy}}{r} = \frac{\bar{r}_{xy}}{\bar{r}} \qquad \left(\frac{q_{xy}}{4q}\right)^2 = 1 \pm \tfrac{1}{4}|q_y|^2 \qquad (2c^\pm)$$

and where the unknowns are the complex functions $q(x, y, t)$ and $r(x, y, t)$, $\alpha_2$ and $\alpha_{-2}$ are given real numbers and overbars mean complex conjugations. Note that for $\alpha_2 = -2$, $\alpha_{-2} = 0$, $(2a^\pm)$ reduces to the non-linear Schrödinger equation $q_t - \mathrm{i}q_{xx} \pm 2\,\mathrm{i}q|q|^2 = 0$ relative to the variables $(x, t)$.

The basic objects of our method of investigation are called asymptotic modules (AM). Generally speaking, an AM is a set of matrix-valued functions $\varphi(k, x, y, \ldots, t)$ with a differentiable structure in $(x, y, \ldots t) \in \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$ and an 'asymptotic' analytic structure in $k$ around some poles $K_1, K_2, \ldots$, on the Riemann sphere. These two structures are coupled thanks to a $\mathscr{D}$-module structure, where $\mathscr{D}$ is some ring of linear differential operators involving $\partial_x, \partial_y, \ldots, \partial_t$ and rational coefficients in $k$. Then some NE connecting some 'asymptotic' coefficients depending on $x, y, \ldots, t$ may arise as a compatibility condition between the differentiable and the analytic structures. AM of the 'same type' lead to the introduction of the same NE for which each of them affords one solution. In 'good' cases these NE are purely differential and involve a small number of unknown functions. The concept of AM is motivated partly by the analysis of the algebraic structure underlying the use of $\bar{\partial}$ (d-bar) equations in [1, 2]. More information on the AM scheme and other applications can be found in [3].

In this letter we will define precisely AM of a special kind, called matrix-bipolar asymptotic modules (MBAM), which involve $2 \times 2$ matrix-valued functions $\phi(k, x, y, t)$ with two poles 0 and $\infty$. We will give $\bar{\partial}$ constructions of MBAM and we will show that MBAM lead us to introduce and to find solutions for an infinite hierarchy of systems of NE including the systems described by (1) and (2). The well known AKSN hierarchy [4] (relative to the variables $(x, t)$) is also recovered as a special case. Each of these systems (S) is purely differential. It consists of $(2+1)$-dimensional evolution equations for at most four complex functions and of quite strong differential constraints (C) similar to $(1c^\pm)$ and $(2c^\pm)$ (note that the last constraint of $(1c^+)$ can be reduced to the sine-Gordon equation $\theta_{xy} = 4\sin\theta$ where $\theta$ is defined by $q_{xy} = 4q\cos\theta$ and $q_y = -2\mathrm{i}\sin\theta$). As a matter of fact, the system (S) describes a time evolution in the manifold of solutions to the bidimensional NE (C). Because of (C), $y$ derivations correspond to some $x$ integrations (for example $(1c^+)$ implies $r_y = \int 4r\,[1+\tfrac{1}{4}(r_x)^2]^{1/2}\,\mathrm{d}x$), so that (S) may be interpreted formally as an integro-differential $(1+1)$-dimensional NE. It should be observed that identifying the variables $x$ and $y$ in MBAM would lead directly to formal integro-differential $(1+1)$-dimensional NE. These $(1+1)$-dimensional NE which enlarge the usual AKNS hierarchy can also be obtained with the inverse scattering transform method by allowing the dispersion relation to be a rational function (see [4] p 265). In some way, using the two variables $x$ and $y$ provides us with a trick for introducing these NE in a correct setting while remaining in a purely differential context.

In this letter proofs are only sketched. A full account of our work will be published elsewhere with additional results such as the existence of local conservation laws and of soliton-like solutions. A generalisation of the present work using matrix multipolar

AM for solving evolution $(N+1)$-dimensional NE with constraints will be presented later.

In order to define a MBAM we introduce the following objects: $U=\{k\}$ is a subset of the Riemann sphere $S$ admitting 0 and $\infty$ as boundary points;

$\Omega=\{(x, y, t)\}$ is some open subset of $\mathbb{R}^3$;

$\mathcal{M}$ is the set of $2\times 2$ matrices with complex entries;

$\mathcal{R}$ is the ring of $\mathcal{M}$-valued functions $\Sigma_{j=-I_1}^{I_0} p_j(x, y, t)\, k^j$ where $I_0$, $I_1\in\mathbb{N}$ and the $p_j$ are $C^\infty$ $\mathcal{M}$-valued functions of $(x, y, t)$;

$f_0(k, x, y, t)=\exp[(-\mathrm{i}kx-\mathrm{i}k^{-1}y+\mathrm{i}\omega(k)t)\sigma_3]$ where $\omega(k)=\Sigma_{j=-J_1}^{J_0}\alpha_j k^j$, $J_0$ and $J_1$ are given integers and the $\alpha_j$ are given complex numbers.

Let $\hat{\mathcal{F}}$ be a set of $\mathcal{M}$-valued functions $\hat{\varphi}(k, x, y, t)$ defined on $U\times\Omega$ and $\mathcal{F}$ be the related set of functions $\varphi(k, x, y, t)=\hat{\varphi}(k, x, y, t)f_0(k, x, y, t)$ with $\hat{\varphi}\in\hat{\mathcal{F}}$. $\mathcal{F}$ is called a MBAM if the following properties are satisfied:

($P_1$) $\hat{\mathcal{F}}$ is a set of $C^\infty$ functions of $(x, y, t)$;

($P_2$) $\hat{\mathcal{F}}$ is a set of 'asymptotic' rational functions of $k$ around 0 and $\infty$, with an 'asymptotic' unit element;

($P_3$) $\mathcal{F}$ is a left $\mathcal{D}$-module, where $\mathcal{D}$ is the ring of linear differential operators generated by $\{\gamma\in\mathbb{R}, \partial_x, \partial_y, \partial_t\}$.

Property ($P_2$) means more precisely that:

(i) any $\hat{\varphi}\in\hat{\mathcal{F}}$ admits asymptotic expansions (AE) around 0 and $\infty$:

$$\hat{\varphi}\underset{\infty}{\sim}\sum_{j=-I_0'}^{\infty}A^jk^{-j}\qquad \hat{\varphi}\underset{0}{\sim}\sum_{j=-I_1'}^{\infty}B^jk^j \tag{3}$$

where $I_0'$ and $I_1'\in\mathbb{N}$ and the coefficients $A^j$ and $B^j$ are $C^\infty$ $\mathcal{M}$-valued functions of $(x, y, t)$;

(ii) the projection

$$\hat{\varphi}\in\hat{\mathcal{F}}\mapsto\widehat{\mathrm{Nor}}\,\hat{\varphi}\doteq\left(\sum_{j=-I_0'}^{0}A^jk^{-j}+\sum_{j=-I_1'}^{-1}B^jk^j\right)\in\mathcal{R}$$

is one-to-one.

(iii) $\hat{\mathcal{F}}$ contains a privileged function $\hat{f}$ such that $\det\hat{f}\neq 0$ and $\widehat{\mathrm{Nor}}\,\hat{f}=I$, i.e.

$$\hat{f}\underset{\infty}{\sim}I+\sum_{j=1}^{\infty}a^jk^{-j}\qquad \hat{f}\underset{0}{\sim}\lambda\left(I+\sum_{j=1}^{\infty}b^jk^j\right) \tag{4}$$

where $I$ is the $2\times 2$ identity matrix and $a^j$, $b^j$ and $\lambda$ are $C^\infty$ $\mathcal{M}$-valued functions of $(x, y, t)$.

We set

$$a^1=\begin{pmatrix}a^{1,1} & a^{1,3}\\ a^{1,2} & a^{1,4}\end{pmatrix}\qquad b^1=\begin{pmatrix}b^{1,1} & b^{1,3}\\ b^{1,2} & b^{1,4}\end{pmatrix}.$$

In the following we outline the proof that the functions $a^{1,2}$, $a^{1,3}$, $b^{1,2}$ and $b^{1,3}$ defined in ($P_2$, iii) satisfy a system of differential NE.

First it is easy to see that $\mathcal{F}$ is a left $\mathcal{R}$-module of dimension one and of basis $f=\hat{f}f_0$, i.e. for any $\varphi\in\mathcal{F}$, $A\doteq\varphi f^{-1}\equiv\hat{\varphi}(\hat{f})^{-1}\in\mathcal{R}$ (see [1] for a similar proof). Furthermore $A=\widehat{\mathrm{Nor}}\,(\hat{\varphi}(\hat{f})^{-1})$. Since $P_3$ implies that $\partial_x f$, $\partial_y f$ and $\partial_t f$ belongs to $\mathcal{F}$, it follows that $(\partial_x f)f^{-1}$, $(\partial_y f)f^{-1}$ and $(\partial_t f)f^{-1}$ belong to $\mathcal{R}$.

In particular, by calculating $\widehat{\mathrm{Nor}}\,((\partial_x f)f^{-1})$ we get

$$(\partial_x+\mathrm{i}k\sigma_3+\mathrm{i}[a^1, \sigma_3])f=0. \tag{5}$$

Equation (4) implies that $\det\hat{f}$ is $1+O(k^{-1})$ as $k\to\infty$ and $\det\lambda+O(k)$ as $k\to 0$. In fact we will suppose that

$$\det\hat{f}=\det\lambda=1. \tag{6}$$

(There is no loss of generality because one can replace the MBAM $\mathscr{F}$ of functions $\varphi$ with the MBAM $\mathscr{F}^r$ of functions $\varphi^r \doteq (\det\hat{f})^{-1/2}\varphi$ for which the corresponding equality (6) is satisfied.) As a consequence of (6) we have $\hat{f}^{-1}=\sigma_2\,{}^t\hat{f}\sigma_2$ (where ${}^t\hat{f}$ is the transpose of $\hat{f}$). By using the AE of $\det\hat{f}$ it is also easy to verify that $a^{1,4}=-a^{1,1}$ and $b^{1,4}=-b^{1,1}$.

On the other hand, since $(\partial_t f)f^{-1}\in\mathscr{R}$, we can identify its AE at 0 and $\infty$ and in particular the coefficients of $k^{-1}$ and $k$. This yields the equations

$$(\partial_t a^1)\sigma_2=-\sum_{j=0}^{J_0}\alpha_j R^{j+1}-\alpha_{-1}\sigma_1+\sum_{j=1}^{J_1}\alpha_{-j}(\lambda S^{j-1}\,{}^t\lambda) \tag{7a}$$

$$(\partial_t b^1)\sigma_2=\sum_{j=1}^{J_0}\alpha_j(\lambda^{-1}R^{j-1}\,{}^t\lambda^{-1})-\sum_{j=0}^{J_1}\alpha_{-j}S^{j+1}-\alpha_1\sigma_1 \tag{7b}$$

where

$$R^j=\begin{pmatrix}R^{j,1} & R^{j,2}\\ R^{j,2} & R^{j,4}\end{pmatrix}\qquad S^j=\begin{pmatrix}S^{j,1} & S^{j,2}\\ S^{j,2} & S^{j,4}\end{pmatrix}$$

are the coefficients of the AE at 0 and $\infty$ of $R(k)\doteq f(k)\sigma_1\,{}^tf(k)$ (we have omitted the $(x,y,t)$ dependence), i.e. $R(k)\underset{\infty}{\sim}\Sigma_{j=0}^{\infty}R^jk^{-j}$ (with $R^0=\sigma_1$), $R(k)\underset{0}{\sim}\lambda(\Sigma_{j=0}^{\infty}S^jk^j)\,{}^t\lambda$ (with $S^0=\sigma_1$).

By using (5) and (6) and reproducing arguments given in [1, 3] it is possible to prove that the entries of $R^j$ are polynomials in $a^{1,2}$ and $a^{1,3}$ and their $x$ derivatives and are computable with the recurrence formulae:

$$R^{j+1,1}=\tfrac{1}{2}\mathrm{i}(R^{j,1})_x+2a^{1,3}R^{j,2}$$

$$R^{j+1,4}=-\tfrac{1}{2}\mathrm{i}(R^{j,4})_x+2a^{1,2}R^{j,2}$$

$$(R^{j,2})_x=-2\mathrm{i}a^{1,2}R^{j,1}+2\mathrm{i}a^{1,3}R^{j,4}.$$

By introducing the new MBAM $\tilde{\mathscr{F}}$ of functions $\tilde{\varphi}(k,y,x,t)\doteq\lambda^{-1}(x,y,t)\varphi(k^{-1},x,y,t)$ whose poles are still 0 and $\infty$ but where the roles of $x$ and $y$ are exchanged, it is shown similarly that the entries of $S^j$ are polynomials in $b^{1,2}$ and $b^{1,3}$ and their $y$ derivatives and are computable by recurrence formulae.

The terms $\lambda S^{j-1}\,{}^t\lambda$ and $\lambda^{-1}R^{j-1}\,{}^t\lambda^{-1}$ in (7a) and (7b) can be obtained from $S^{j-1}$, $R^{j-1}$ and the products $\lambda^\mu\lambda^\nu$ ($\mu,\nu\in\{1,2,3,4\}$) of the entries of

$$\lambda=\begin{pmatrix}\lambda^1 & \lambda^3\\ \lambda^2 & \lambda^4\end{pmatrix}.$$

In order to evaluate these products we use the fact that $(\partial_x f)f^{-1}\in\mathscr{R}$. By identifying its AE at 0 and $\infty$, and in particular the coefficients of $k$ and $k^0$, we get

$${}^t\lambda\sigma_1\lambda=\sigma_1+\sigma_2(b^1)_x \tag{8a}$$

$$[a^1,\sigma_3]=\mathrm{i}\lambda_x\sigma_2\,{}^t\lambda\sigma_2. \tag{8b}$$

Since $(\partial_y f)f^{-1}\in\mathscr{R}$, we find in an analogous way

$$\lambda\sigma_1\,{}^t\lambda=\sigma_1-(a^1)_y\sigma_2 \tag{8c}$$

$$[b^1,\sigma_3]=\mathrm{i}\sigma_2\,{}^t\lambda_y\sigma_2\lambda. \tag{8d}$$

Equations (6) and (8) yield

$$\lambda^1\lambda^3 = -\tfrac{1}{2}\mathrm{i}(a^{1,3})_y \qquad \lambda^2\lambda^3 = -\tfrac{1}{2} - \tfrac{1}{8}(a^{1,2})_{xy}/a^{1,2}$$
$$(\lambda^1)^2 = (-\tfrac{1}{2} + \tfrac{1}{8}(a^{1,2})_{xy}/a^{1,2})(b^{1,2})_x/(a^{1,2})_y$$

and similar expressions for the other terms $\lambda^\mu\lambda^\nu$. As a byproduct we obtain the constraints

$$(a^{1,2})_y(a^{1,3})_y = (b^{1,2})_x(b^{1,3})_x \tag{9a}$$

$$[(a^{1,2})_{xy}/4a^{1,2}]^2 = 1 + (a^{1,2})_y(a^{1,3})_y \tag{9b}$$

$$(a^{1,2})_{xy}/a^{1,2} = (a^{1,3})_{xy}/a^{1,3} = (b^{1,2})_{xy}/b^{1,2} = (b^{1,3})_{xy}/b^{1,3}. \tag{9c}$$

From the previous analysis it follows that equations ($7a$) and ($7b$) yield purely differential relations between $a^{1,2}$, $a^{1,3}$, $b^{1,2}$ and $b^{1,3}$. Thus we obtain a system (S) of NE consisting of four scalar differential evolution NE and of the differential constraints (9) with four unknown scalar functions $a^{1,2}$, $a^{1,3}$, $b^{1,2}$ and $b^{1,3}$ depending on the variables $(x, y, t)$. Any previously defined MBAM $\mathscr{F}$ affords one solution to the system (S). By varying $J_0$, $J_1$ and the $\alpha_j$ in the expression for $\omega(k)$ we generate a hierarchy (H) of solvable systems (S).

If $\alpha_j = 0$ for $j < 0$, then (H) leads to the AKNS hierarchy [4] (in the variables $(x, t)$) for the functions $q_1 \doteq 2\mathrm{i}a_1^3$ and $q_2 \doteq -2\mathrm{i}a_1^2$.

If all the $\alpha_j$ are zero except $\alpha_3$ and $\alpha_{-3}$ and $f(k) = \sigma_1 f(-k)\sigma_1$ (resp $f(k) = \sigma_2 f(-k)\sigma_2$) then $a^{1,2} = -a^{1,3}$, $b^{1,2} = -b^{1,3}$ (resp $a^{1,2} = a^{1,3}$, $b^{1,2} = b^{1,3}$) and (S) yields the system ($1a^+$, $b^+$, $c^+$) (resp ($1a^-$, $b^-$, $c^-$)) for the functions $q \doteq 2\mathrm{i}a^{1,3}$ and $r \doteq 2\mathrm{i}b^{1,3}$.

If all the $\alpha_j$ are zero except $\alpha_2$ and $\alpha_{-2}$ which are real and $f(k) = \sigma_1 \overline{f(\bar{k})}\sigma_1$ (resp $f(k) = \sigma_2 \overline{f(\bar{k})}\sigma_2$), then $a^{1,2} = \overline{a^{1,3}}$, $b^{1,2} = \overline{b^{1,3}}$ (resp $a^{1,2} = -\overline{a^{1,3}}$, $b^{1,2} = -\overline{b^{1,3}}$) and (S) yields the system ($2a^+$, $b^+$, $c^+$) (resp ($2a^-$, $b^-$, $c^-$)) for the functions $q \doteq 2\mathrm{i}a^{1,3}$ and $r \doteq 2\mathrm{i}b^{1,3}$.

MBAM can be constructed as follows. Given a distribution

$$r(k) = \begin{pmatrix} 0 & r_1(k) \\ r_2(k) & 0 \end{pmatrix} \qquad (k \in \mathbb{C} - \{0\})$$

going to zero fast enough as $k \to \infty$ and $k \to 0$, we consider the set $\mathscr{F}$ of $\mathscr{M}$-valued functions $\varphi(k, x, y, t)$ which satisfy the $\bar{\partial}$ equation

$$\frac{\partial\varphi}{\partial\bar{k}}(k) = \varphi(k)r(k) \qquad k \in \mathbb{C} - \{0\} \tag{10}$$

and are such that the functions $\hat{\varphi} \doteq \varphi(f_0)^{-1}$ admit AE in the form (3) at 0 and $\infty$. By applying the generalised Cauchy formula in a way similar to [1, 2] one can see that $\hat{\varphi}$ is a solution of the integral equation $(I - J)\hat{\varphi} = \widehat{\mathrm{Nor}}\ \hat{\varphi}$ where

$$J\hat{\varphi}(k) = \frac{1}{2\mathrm{i}\pi}\int\int_{\mathbb{R}^2} \frac{\mathrm{d}q \wedge \mathrm{d}\bar{q}}{q - k}\,\hat{\varphi}(q) f_0(q) r(q) f_0^{-1}(q). \tag{11}$$

With reasonable assumptions on $r(k)$ it is easy to verify that $\mathscr{F}$ is a MBAM. We remark that (10) implies $\partial_{\bar{k}}(\det\hat{f}) = 0$ so that condition (6) is satisfied.

Note that for $\mathrm{i} = 1, 2$ the reduction $r(k) = -\sigma_i\ r(-k)\sigma_i$ (resp $r(k) = \sigma_i\ \overline{r(\bar{k})}\ \sigma_i$) corresponds to the reduction $f(k) = \sigma_i f(-k)\sigma_i$ (resp $f(k) = \sigma_i \overline{f(\bar{k})}\sigma_i$).

If $r(k)$ is a linear combination of delta functions the associated solution of the system (S) can be computed easily from (11).

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